

Locally Anisotropic Black Holes in Einstein Gravity

Sergiu I. Vacaru ^{*}

Institute of Applied Physics, Academy of Sciences,
Academy str. 5, Chișinău MD2028,
Republic of Moldova

(December 31, 1999; version 4: May 8, 2000)

PACS: 04.20.Jb, 04.70.Bw **gr-qc/0001020**

Abstract

By applying the method of moving frames modelling one and two dimensional local anisotropies we construct new solutions of Einstein equations on pseudo-Riemannian spacetimes. The first class of solutions describes non-trivial deformations of static spherically symmetric black holes to locally anisotropic ones which have elliptic (in three dimensions) and ellipsoidal, toroidal and elliptic and another forms of cylinder symmetries (in four dimensions). The second class consists from black holes with oscillating elliptic horizons.

^{*}© Sergiu I. Vacaru, e-mail: vacaru@lises.asm.md

1 Introduction

In recent years, there has been great interest in investigation of gravitational models with anisotropies and applications in modern cosmology and astrophysics. There are possible locally anisotropic inflational and black hole like solutions of Einstein equations in the framework of so-called generalized Finsler–Kaluza–Klein models [9] and in low-energy locally anisotropic limits of (super) string theories [10].

In this paper we shall restrict ourselves to a more limited problem of definition of black hole solutions with local anisotropy in the framework of the Einstein theory (in three and four dimensions). Our purpose is to construct solutions of gravitational field equations by imposing symmetries differing in appearance from the static spherical one (which uniquely results in the Schwarzschild solution) and search for solutions with configurations of event horizons like rotation ellipsoids, torus and ellipsoidal and cylinders. We shall proof that there are possible elliptic oscillations in time of horizons.

In order to simplify the procedure of solution and investigate more deeply the physical implications of general relativistic models with local anisotropy we shall transfer our analysis with respect to anholonomic frames which are equivalently characterized by nonlinear connection (N -connection) structures [2, 3, 8, 9, 10]. This geometric approach is very useful for construction of metrics with prescribed symmetries of horizons and definition of conditions when such type black hole like solutions could be selected from an integral variety of the Einstein field equations with a corresponding energy-momentum tensor. We argue that, in general, the symmetries of solutions are not completely determined by the field equations and coordinate conditions but there are also required some physical motivations for choosing of corresponding classes of systems of reference (prescribed type of local anisotropy and symmetries of horizons) with respect to which the ‘picture’ of interactions, symmetries and conservation laws is drawn in the simplest form.

The paper is organized as follows: In section 2 we introduce metrics and anholonomic frames with local anisotropies admitting equivalent N -connection structures. We write down the Einstein equations with respect to such locally anisotropic frames. In section 3 we analyze the general properties of the system of gravitational field equations for an ansatz for metrics with local anisotropy. In section 4 we generalize the three dimensional static black hole solution to the case with elliptic horizon and proof that there are possible elliptic oscillations in time of locally anisotropic black holes. The section 5 is devoted to four dimensional locally anisotropic static solutions with rotation ellipsoidal, toroidal and cylindrical like horizons and consider elliptic oscillations in time. In the last section we make some final remarks.

2 Anholonomic frames and N–connections

In this section we outline the necessary results on spacetime differential geometry [4] and anholonomic frames induced by N–connection structures [8, 9, 10]. We examine an ansatz for locally anisotropic (pseudo) Riemannian metrics with respect to coordinate bases and illustrate a substantial geometric simplification and reduction of the number of coefficients of geometric objects and field equations after linear transforms to anholonomic bases defined by coefficients of a corresponding N–connection. The Einstein equations are rewritten in an invariant form with respect to such locally anisotropic bases.

Consider a class of pseudo–Riemannian metrics

$$g = g_{\underline{\alpha}\underline{\beta}}(u^\varepsilon) \ du^{\underline{\alpha}} \otimes du^{\underline{\beta}}$$

in a $n+m$ dimensional spacetime $V^{(n+m)}$, ($n=2$ and $m=1, 2$), with components

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}, \quad (2.1)$$

where $g_{ij} = g_{ij}(u^\alpha)$ and $h_{ab} = h_{ab}(u^\alpha)$ are respectively some symmetric $n \times n$ and $m \times m$ dimensional matrices, $N_j^e = N_j^e(u^\beta)$ is a $n \times m$ matrix, and the $n+m$ dimensional local coordinates are provided with general Greek indices and denoted $u^\beta = (x^i, y^a)$. The Latin indices i, j, k, \dots in (2.1) run values 1, 2 and a, b, c, \dots run values 3, 4 and we note that both type of isotropic, x^i , and the so–called anisotropic, y^a , coordinates could be space or time like ones. We underline indices in order to emphasize that components are given with respect to a coordinate (holonomic) basis

$$e_{\underline{\alpha}} = \partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}} \quad (2.2)$$

and/or its dual

$$e^{\underline{\alpha}} = du^{\underline{\alpha}}. \quad (2.3)$$

The class of metrics (2.1) transform into a $(n \times n) \oplus (m \times m)$ block form

$$g = g_{ij}(u^\varepsilon) \ dx^i \otimes dx^j + h_{ab}(u^\varepsilon) (\delta y^a)^2 \otimes (\delta y^a)^2 \quad (2.4)$$

if one chooses a frame of basis vectors

$$\delta_\alpha = \delta/\partial u^\alpha = (\delta/\partial x^i = \partial_i - N_i^a(u^\varepsilon) \partial_a, \partial_b), \quad (2.5)$$

where $\partial_i = \partial/x^i$ and $\partial_a = \partial/\partial y^a$, with the dual basis being

$$\delta^\alpha = \delta u^\alpha = (dx^i, \delta y^a = dy^a + N_i^a(u^\varepsilon) dx^i). \quad (2.6)$$

The set of coefficients $N = \{N_i^a(u^\varepsilon)\}$ from (2.5) and (2.6) could be associated to components of a nonlinear connection (in brief, N–connection) structure defining a local decomposition of spacetime into n isotropic directions x^i

and one or two anisotropic directions y^a . The global definition of N-connection is due to W. Barthel [2] (the rigorous mathematical definition of N-connection is possible on the language of exact sequences of vector, or tangent, subbundles) and this concept is largely applied in Finsler geometry and its generalizations [3, 8]. It was concluded [9, 10] that N-connection structures are induced under non-trivial dynamical compactifications of higher dimensions in (super) string and (super) gravity theories and even in general relativity if we are dealing with anholonomic frames.

A N-connection is characterized by its curvature, N-curvature,

$$\Omega_{ij}^a = \partial_i N_j^a - \partial_j N_i^a + N_i^b \partial_b N_j^a - N_j^b \partial_b N_i^a. \quad (2.7)$$

As a particular case we obtain a linear connection field $\Gamma_{ib}^a(x^i)$ if $N_i^a(x^i, y^a) = \Gamma_{ib}^a(x^i, y^a)$ [8, 9].

For nonvanishing values of Ω_{ij}^a the basis (2.5) is anholonomic and satisfies the conditions

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma_{\alpha\beta} \delta_\gamma,$$

where the anholonomy coefficients $w^\gamma_{\alpha\beta}$ are defined by the components of N-connection,

$$\begin{aligned} w_{ij}^k &= 0, w_{aj}^k = 0, w_{ia}^k = 0, w_{ab}^k = 0, w_{ab}^c = 0, \\ w_{ij}^a &= -\Omega_{ij}^a, w_{aj}^b = -\partial_a N_j^b, w_{ia}^b = \partial_a N_i^b. \end{aligned}$$

We emphasize that the elongated by N-connection operators (2.5) and (2.6) must be used, respectively, instead of local operators of partial derivation (2.2) and differentials (2.3) if some differential calculations are performed with respect to any anholonomic bases locally adapted to a fixed N-connection structure (in brief, we shall call such local frames as la-bases or la-frames, where, in brief, la- is from locally anisotropic).

The torsion, $T(\delta_\gamma, \delta_\beta) = T_{\beta\gamma}^\alpha \delta_\alpha$, and curvature, $R(\delta_\tau, \delta_\gamma) \delta_\beta = R_{\beta\gamma}^\alpha \delta_\alpha$, tensors of a linear connection $\Gamma_{\beta\gamma}^\alpha$ are introduced in a usual manner and, respectively, have the components

$$T_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \Gamma_{\gamma\beta}^\alpha + w_{\beta\gamma}^\alpha \quad (2.8)$$

and

$$R_{\beta\gamma}^\alpha = \delta_\tau \Gamma_{\beta\gamma}^\alpha - \delta_\gamma \Gamma_{\beta\delta}^\alpha + \Gamma_{\beta\gamma}^\varphi \Gamma_{\varphi\tau}^\alpha - \Gamma_{\beta\tau}^\varphi \Gamma_{\varphi\gamma}^\alpha + \Gamma_{\beta\varphi}^\alpha w_{\gamma\tau}^\varphi. \quad (2.9)$$

The Ricci tensor is defined

$$R_{\beta\gamma} = R_{\beta\gamma}^\alpha \quad (2.10)$$

and the scalar curvature is

$$R = g^{\beta\gamma} R_{\beta\gamma}. \quad (2.11)$$

The Einstein equations with respect to a la-basis (2.6) are written

$$R_{\beta\gamma} - \frac{R}{2} g_{\beta\gamma} = k \Upsilon_{\beta\gamma}, \quad (2.12)$$

where the energy-momentum d-tensor $\Upsilon_{\beta\gamma}$ includes the cosmological constant terms and possible contributions of torsion (2.8) and matter and k is the coupling constant. For a symmetric linear connection the torsion field can be considered as induced by the anholonomy coefficients. For dynamical torsions there are necessary additional field equations, see, for instance, the case of locally anisotropic gauge like theories [11].

The geometrical objects with respect to a la-bases are distinguished by the corresponding N-connection structure and called (in brief) d-tensors, d-metrics (2.4), linear d-connections and so on [8, 9, 10].

A linear d-connection D on a spacetime V ,

$$D_{\delta\gamma}\delta_\beta = \Gamma^\alpha_{\beta\gamma}(x^k, y^a)\delta_\alpha,$$

is parametrized by non-trivial horizontal (isotropic) – vertical (anisotropic), in brief, h-v-components,

$$\Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}). \quad (2.13)$$

Some d-connection and d-metric structures are compatible if there are satisfied the conditions

$$D_\alpha g_{\beta\gamma} = 0.$$

For instance, the canonical compatible d-connection

$${}^c\Gamma^\alpha_{\beta\gamma} = ({}^cL^i_{jk}, {}^cL^a_{bk}, {}^cC^i_{jc}, {}^cC^a_{bc})$$

is defined by the coefficients of d-metric (2.4), $g_{ij}(x^i, y^a)$ and $h_{ab}(x^i, y^a)$, and of N-connection, $N_i^a = N_i^a(x^i, y^b)$,

$$\begin{aligned} {}^cL^i_{jk} &= \frac{1}{2}g^{in}(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}), \\ {}^cL^a_{bk} &= \partial_b N_k^a + \frac{1}{2}h^{ac}(\delta_k h_{bc} - h_{dc}\partial_b N_i^d - h_{db}\partial_c N_i^d), \\ {}^cC^i_{jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk}, \\ {}^cC^a_{bc} &= \frac{1}{2}h^{ad}(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}). \end{aligned} \quad (2.14)$$

The coefficients of the canonical d-connection generalize with respect to la-bases the well known Cristoffel symbols.

For a d-connection (2.13) we can compute the non-trivial components of d-torsion (2.8)

$$\begin{aligned} T^i_{jk} &= T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = C^i_{ja}, \quad T^i_{aj} = -C^i_{ja}, \\ T^i_{ja} &= 0, \quad T^a_{bc} = S^a_{bc} = C^a_{bc} - C^a_{cb}, \\ T^a_{ij} &= -\Omega^a_{ij}, \quad T^a_{bi} = \partial_b N_i^a - L^a_{bj}, \quad T^a_{ib} = -T^a_{bi}. \end{aligned} \quad (2.15)$$

In a similar manner, putting non-vanishing coefficients (2.13) into the formula for curvature (2.9), we can compute the coefficients of d-curvature

$$R(\delta_\tau, \delta_\gamma) \delta_\beta = R_{\beta}^{\alpha}{}_{\gamma\tau} \delta_\alpha,$$

split into h-, v-invariant components,

$$\begin{aligned} R_{h,jk}^i &= \delta_k L_{.hj}^i - \delta_j L_{.hk}^i + L_{.hj}^m L_{mk}^i - L_{.hk}^m L_{mj}^i - C_{.ha}^i \Omega_{.jk}^a, \\ R_{b,jk}^a &= \delta_k L_{.bj}^a - \delta_j L_{.bk}^a + L_{.bj}^c L_{.ck}^a - L_{.bk}^c L_{.cj}^a - C_{.bc}^a \Omega_{.jk}^c, \\ P_{j,ka}^i &= \partial_k L_{.jk}^i + C_{.jb}^i T_{.ka}^b - (\partial_k C_{.ja}^i + L_{.lk}^i C_{.ja}^l - L_{.jk}^l C_{.la}^i - L_{.ak}^c C_{.jc}^i), \\ P_{b,ka}^c &= \partial_a L_{.bk}^c + C_{.bd}^c T_{.ka}^d - (\partial_k C_{.ba}^c + L_{.dk}^c C_{.ba}^d - L_{.bk}^d C_{.da}^c - L_{.ak}^d C_{.bd}^c) \\ S_{j,bc}^i &= \partial_c C_{.jb}^i - \partial_b C_{.jc}^i + C_{.jb}^h C_{.hc}^i - C_{.jc}^h C_{.hb}^i, \\ S_{b,cd}^a &= \partial_d C_{.bc}^a - \partial_c C_{.bd}^a + C_{.bc}^e C_{.ed}^a - C_{.bd}^e C_{.ec}^a. \end{aligned}$$

The components of the Ricci tensor (2.10) with respect to locally adapted frames (2.5) and (2.6) (in this case, d-tensor) are as follows:

$$\begin{aligned} R_{ij} &= R_{i,jk}^k, \quad R_{ia} = {}^2 P_{ia} = -P_{i,ka}^k, \\ R_{ai} &= {}^1 P_{ai} = P_{a,ib}^b, \quad R_{ab} = S_{a,bc}^c. \end{aligned} \tag{2.16}$$

We point out that because, in general, ${}^1 P_{ai} \neq {}^2 P_{ia}$ the Ricci d-tensor is non symmetric. This is a consequence of anholonomy of la-bases.

Having defined a d-metric of type (2.4) on spacetime V we can compute the scalar curvature (2.11) of a d-connection D ,

$$\overline{R} = G^{\alpha\beta} R_{\alpha\beta} = \hat{R} + S, \tag{2.17}$$

where $\hat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

Now, by introducing the values of (2.16) and (2.17) into equations (2.12), the Einstein equations with respect to a la-basis seen to be

$$\begin{aligned} R_{ij} - \frac{1}{2} (\hat{R} + S) g_{ij} &= k \Upsilon_{ij}, \\ S_{ab} - \frac{1}{2} (\hat{R} + S) h_{ab} &= k \Upsilon_{ab}, \\ {}^1 P_{ai} &= k \Upsilon_{ai}, \\ {}^2 P_{ia} &= -k \Upsilon_{ia}, \end{aligned} \tag{2.18}$$

where Υ_{ij} , Υ_{ab} , Υ_{ai} and Υ_{ia} are the components of the energy-momentum d-tensor field $\Upsilon_{\beta\gamma}$ (which includes possible cosmological constants, contributions of anholonomy d-torsions (2.15) and matter) and k is the coupling constant. For simplicity, we omitted the upper left index c pointing that for the Einstein theory the Ricci d-tensor and curvature scalar should be computed by applying the coefficients of canonical d-connection (2.14).

3 An ansatz for la-metrics

Let us consider a four dimensional (in brief, 4D) spacetime $V^{(2+2)}$ (with two isotropic plus two anisotropic local coordinates) provided with a metric (2.1) (of signature $(-, +, +, +)$, or $(+, +, +, -)$, $(+, +, -, +)$) parametrized by a symmetric matrix of type

$$\begin{bmatrix} g_1 + q_1^2 h_3 + n_1^2 h_4 & 0 & q_1 h_3 & n_1 h_4 \\ 0 & g_2 + q_2^2 h_3 + n_2^2 h_4 & q_2 h_3 & n_2 h_4 \\ q_1 h_3 & q_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{bmatrix} \quad (3.1)$$

with components being some functions

$$g_i = g_i(x^j), q_i = q_i(x^j, z), n_i = n_i(x^j, z), h_a = h_a(x^j, z)$$

of necessary smoothly class. With respect to a la-basis (2.6) this ansatz results in diagonal 2×2 h- and v-metrics for a d-metric (2.6) (for simplicity, we shall consider only diagonal 2D nondegenerated metrics because for such dimensions every symmetric matrix can be diagonalized).

An equivalent diagonal d-metric (2.4) is obtained for the associated N-connection with coefficients being functions on three coordinates (x^i, z) ,

$$\begin{aligned} N_1^3 &= q_1(x^i, z), & N_2^3 &= q_2(x^i, z), \\ N_1^4 &= n_1(x^i, z), & N_2^4 &= n_2(x^i, z). \end{aligned} \quad (3.2)$$

For simplicity, we shall use brief denotations of partial derivatives, like $\dot{a} = \partial a / \partial x^1$, $a' = \partial a / \partial x^2$, $a^* = \partial a / \partial z$, $\dot{a}' = \partial^2 a / \partial x^1 \partial x^2$, $a^{**} = \partial^2 a / \partial z \partial z$.

The non-trivial components of the Ricci d-tensor (2.16) (for the ansatz (3.1)) when $R_1^1 = R_2^2$ and $S_3^3 = S_4^4$, are computed

$$R_1^1 = \frac{1}{2g_1 g_2} [-(g_1'' + \ddot{g}_2) + \frac{1}{2g_2} (\dot{g}_2^2 + g'_1 g'_2) + \frac{1}{2g_1} (g'_1{}^2 + \dot{g}_1 \dot{g}_2)], \quad (3.3)$$

$$S_3^3 = \frac{1}{h_3 h_4} [-h_4^{**} + \frac{1}{2h_4} (h_4^*)^2 + \frac{1}{2h_3} h_3^* h_4^*], \quad (3.4)$$

$$\begin{aligned} P_{3i} &= \frac{q_i}{2} \left[\left(\frac{h_3^*}{h_3} \right)^2 - \frac{h_3^{**}}{h_3} + \frac{h_4^*}{2h_4^2} - \frac{h_3^* h_4^*}{2h_3 h_4} \right] \\ &\quad + \frac{1}{2h_4} \left[\frac{\dot{h}_4}{2h_4} h_4^* - \dot{h}_4^* + \frac{\dot{h}_3}{2h_3} h_4^* \right], \end{aligned} \quad (3.5)$$

$$P_{4i} = -\frac{h_4}{2h_3} n_i^{**}. \quad (3.6)$$

The curvature scalar \overleftarrow{R} (2.17) is defined by two non-trivial components $\hat{R} = 2R_1^1$ and $S = 2S_3^3$.

The system of Einstein equations (2.18) transforms into

$$R_1^1 = -\kappa \Upsilon_3^3 = -\kappa \Upsilon_4^4, \quad (3.7)$$

$$S_3^3 = -\kappa \Upsilon_1^1 = -\kappa \Upsilon_2^2, \quad (3.8)$$

$$P_{3i} = \kappa \Upsilon_{3i}, \quad (3.9)$$

$$P_{4i} = \kappa \Upsilon_{4i}, \quad (3.10)$$

where the values of R_1^1, S_3^3, P_{ai} , are taken respectively from (3.3), (3.4), (3.5), (3.6).

We note that we can define the N-coefficients (3.2), $q_i(x^k, z)$ and $n_i(x^k, z)$, by solving the equations (3.9) and (3.10) if the functions $h_i(x^k, z)$ are known as solutions of the equations (3.8).

Let us analyze the basic properties of equations (3.8)–(3.10) (the h-equations will be considered for 3D and 4D in the next sections). The v-component of the Einstein equations (3.7)

$$\frac{\partial^2 h_4}{\partial z^2} - \frac{1}{2h_4} \left(\frac{\partial h_4}{\partial z} \right)^2 - \frac{1}{2h_3} \left(\frac{\partial h_3}{\partial z} \right) \left(\frac{\partial h_4}{\partial z} \right) - \frac{\kappa}{2} \Upsilon_1 h_3 h_4 = 0 \quad (3.11)$$

(here we write down the partial derivatives on z in explicit form) follows from (3.4) and (3.8) and relates some first and second order partial on z derivatives of diagonal components $h_a(x^i, z)$ of a v-metric with a source $\kappa \Upsilon_1(x^i, z) = \kappa \Upsilon_1^1 = \kappa \Upsilon_2^2$ in the h-subspace. We can consider as unknown the function $h_3(x^i, z)$ (or, inversely, $h_4(x^i, z)$) for some compatible values of $h_4(x^i, z)$ (or $h_3(x^i, z)$) and source $\Upsilon_1(x^i, z)$.

By introducing a new variable $\beta = h_4^*/h_4$ the equation (3.11) transforms into

$$\beta^* + \frac{1}{2}\beta^2 - \frac{\beta h_3^*}{2h_3} - 2\kappa \Upsilon_1 h_3 = 0 \quad (3.12)$$

which relates two functions $\beta(x^i, z)$ and $h_3(x^i, z)$. There are two possibilities: 1) to define β (i. e. h_4) when $\kappa \Upsilon_1$ and h_3 are prescribed and, inversely 2) to find h_3 for given $\kappa \Upsilon_1$ and h_4 (i. e. β); in both cases one considers only “**” derivatives on z -variable (coordinates x^i are treated as parameters).

1. In the first case the explicit solutions of (3.12) have to be constructed by using the integral varieties of the general Riccati equation [6] which by a corresponding redefinition of variables, $z \rightarrow z(\varsigma)$ and $\beta(z) \rightarrow \eta(\varsigma)$ (for simplicity, we omit here the dependencies on x^i) could be written in the canonical form

$$\frac{\partial \eta}{\partial \varsigma} + \eta^2 + \Psi(\varsigma) = 0$$

where Ψ vanishes for vacuum gravitational fields. In vacuum cases the Riccati equation reduces to a Bernoulli equation which (we can use the former variables) for $s(z) = \beta^{-1}$ transforms into a linear differential (on z) equation,

$$s^* + \frac{h_3^*}{2h_3}s - \frac{1}{2} = 0. \quad (3.13)$$

2. In the second (inverse) case when h_3 is to be found for some prescribed $\kappa\Upsilon_1$ and β the equation (3.12) is to be treated as a Bernoulli type equation,

$$h_3^* = -\frac{4\kappa\Upsilon_1}{\beta}(h_3)^2 + \left(\frac{2\beta^*}{\beta} + \beta\right)h_3 \quad (3.14)$$

which can be solved by standard methods. In the vacuum case the squared on h_3 term vanishes and we obtain a linear differential (on z) equation.

A particular interest presents those solutions of the equation (3.12) which via 2D conformal transforms with a factor $\omega = \omega(x^i, z)$ are equivalent to a diagonal h-metric on x -variables, i.e. one holds the parametrization

$$h_3 = \omega(x^i, z) a_3(x^i) \text{ and } h_4 = \omega(x^i, z) a_4(x^i), \quad (3.15)$$

where $a_3(x^i)$ and $a_4(x^i)$ are some arbitrary functions (for instance, we can impose the condition that they describe some 2D soliton like or black hole solutions). In this case $\beta = \omega^*/\omega$ and for $\gamma = \omega^{-1}$ the equation (3.12) transforms into

$$\gamma \gamma^{**} = -2\kappa\Upsilon_1 a_3(x^i) \quad (3.16)$$

with the integral variety determined by

$$z = \int \frac{d\gamma}{\sqrt{|-4k\Upsilon_1 a_3(x^i) \ln|\gamma| + C_1(x^i)|}} + C_2(x^i), \quad (3.17)$$

where it is considered that the source Υ_1 does not depend on z .

Finally, we conclude that the v-metrics are defined by the integral varieties of corresponding Riccati and/or Bernoulli equations with respect to z -variables with the h-coordinates x^i treated as parameters.

4 3D black la-holes

Let us analyze some basic properties of 3D spacetimes $V^{(2+1)}$ (we emphasize that in approach $(2+1)$ points to a splitting into two isotropic and one anisotropic directions and not to usual 2D space plus one time like coordinates; in general anisotropies could be associate to both space and/or time like coordinates) provided with d-metrics of type

$$\delta s^2 = g_1(x^k)(dx^1)^2 + g_2(x^k)(dx^2)^2 + h_3(x^i, z)(\delta z)^2, \quad (4.1)$$

where x^k are 2D coordinates, $y^3 = z$ is the anisotropic coordinate and

$$\delta z = dz + N_i^3(x^k, z)dx^i.$$

The N-connection coefficients are

$$N_1^3 = q_1(x^i, z), \quad N_2^3 = q_2(x^i, z). \quad (4.2)$$

The non-trivial components of the Ricci d-tensor (2.16), for the ansatz (3.1) with $h_4 = 1$ and $n_i = 0$, $R_1^1 = R_2^2$ and P_{3i} , are

$$R_1^1 = \frac{1}{2g_1g_2} [-(g_1'' + \ddot{g}_2) + \frac{1}{2g_2} (\dot{g}_2^2 + g'_1g'_2) + \frac{1}{2g_1} (g_1'^2 + \dot{g}_1\dot{g}_2)], \quad (4.3)$$

$$P_{3i} = \frac{q_i}{2} \left[\left(\frac{h_3^*}{h_3} \right)^2 - \frac{h_3^{**}}{h_3} \right] \quad (4.4)$$

(for 3D the component $\overset{\leftrightarrow}{S}_3^3 \equiv 0$, see (3.4)).

The curvature scalar $\overset{\leftrightarrow}{R}$ (2.17) is $\overset{\leftrightarrow}{R} = \hat{R} = 2R_1^1$.

The system of Einstein equations (2.18) transforms into

$$R_1^1 = -\kappa \Upsilon_3^3, \quad (4.5)$$

$$P_{3i} = \kappa \Upsilon_{3i}, \quad (4.6)$$

which is compatible for energy-momentum d-tensors with $\Upsilon_1^1 = \Upsilon_2^2 = 0$; the values of R_1^1 and P_{3i} are taken respectively from (4.3) and (4.4).

By using the equation (4.6) we can define the N-coefficients (4.2), $q_i(x^k, z)$, if the function $h_3(x^k, z)$ and the components Υ_{3i} of the energy-momentum d-tensor are given. We note that the equations (4.4) are solved for arbitrary functions $h_3 = h_3(x^k)$ and $q_i = q_i(x^k, z)$ if $\Upsilon_{3i} = 0$ and in this case the component of d-metric $h_3(x^k)$ is not contained in the system of 3D field equations.

4.1 Static elliptic horizons

Let us consider a class of 3D d-metrics which local anisotropy which are similar to Banados–Teitelboim–Zanelli (BTZ) black holes [1].

The d-metric is parametrized

$$\delta s^2 = g_1(\chi^1, \chi^2) (d\chi^1)^2 + (d\chi^2)^2 - h_3(\chi^1, \chi^2, t) (\delta t)^2, \quad (4.7)$$

where $\chi^1 = r/r_h$ for $r_h = const$, $\chi^2 = \theta/r_a$ if $r_a = \sqrt{|\kappa \Upsilon_3^3|} \neq 0$ and $\chi^2 = \theta$ if $\Upsilon_3^3 = 0$, $y^3 = z = t$, where t is the time like coordinate. The Einstein equations (4.5) and (4.6) transforms respectively into

$$\frac{\partial^2 g_1}{\partial(\chi^2)^2} - \frac{1}{2g_1} \left(\frac{\partial g_1}{\partial \chi^2} \right)^2 - 2\kappa \Upsilon_3^3 g_1 = 0 \quad (4.8)$$

and

$$\left[\frac{1}{h_3} \frac{\partial^2 h_3}{\partial z^2} - \left(\frac{1}{h_3} \frac{\partial h_3}{\partial z} \right)^2 \right] q_i = -\kappa \Upsilon_{3i}. \quad (4.9)$$

By introducing new variables

$$p = g'_1/g_1 \text{ and } s = h_3^*/h_3 \quad (4.10)$$

where the 'prime' in this subsection denotes the partial derivative ∂/χ^2 , the equations (4.8) and (4.9) transform into

$$p' + \frac{p^2}{2} + 2\epsilon = 0 \quad (4.11)$$

and

$$s^* q_i = \kappa \Upsilon_{3i}, \quad (4.12)$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = 1(-1)$ for the signature $1(-1)$ of the anisotropic coordinate.

A class of solutions of 3D Einstein equations for arbitrary $q_i = q_i(\chi^k, t)$ and $\Upsilon_{3i} = 0$ is obtained if $s = s(\chi^i)$. After integration of the second equation from (4.10), we find

$$h_3(\chi^k, t) = h_{3(0)}(\chi^k) \exp [s_{(0)}(\chi^k) t] \quad (4.13)$$

as a general solution of the system (4.12) with vanishing right part. Static solutions are stipulated by $q_i = q_i(\chi^k)$ and $s_{(0)}(\chi^k) = 0$.

The integral curve of (4.11), intersecting a point $(\chi_{(0)}^2, p_{(0)})$, considered as a differential equation on χ^2 is defined by the functions [6]

$$p = \frac{p_{(0)}}{1 + \frac{p_{(0)}}{2} (\chi^2 - \chi_{(0)}^2)}, \quad \epsilon = 0; \quad (4.14)$$

$$p = \frac{p_{(0)} - 2 \tanh(\chi^2 - \chi_{(0)}^2)}{1 + \frac{p_{(0)}}{2} \tanh(\chi^2 - \chi_{(0)}^2)}, \quad \epsilon > 0; \quad (4.15)$$

$$p = \frac{p_{(0)} - 2 \tan(\chi^2 - \chi_{(0)}^2)}{1 + \frac{p_{(0)}}{2} \tan(\chi^2 - \chi_{(0)}^2)}, \quad \epsilon < 0. \quad (4.16)$$

Because the function p depends also parametrically on variable χ^1 we must consider functions $\chi_{(0)}^2 = \chi_{(0)}^2(\chi^1)$ and $p_{(0)} = p_{(0)}(\chi^1)$.

For simplicity, here we elucidate the case $\epsilon < 0$. The general formula for the nontrivial component of h-metric is to be obtained after integration on χ^1 of (4.16) (see formula (4.10))

$$g_1(\chi^1, \chi^2) = g_{1(0)}(\chi^1) \left\{ \sin[\chi^2 - \chi_{(0)}^2(\chi^1)] + \arctan \frac{2}{p_{(0)}(\chi^1)} \right\}^2,$$

for $p_{(0)}(\chi^1) \neq 0$, and

$$g_1(\chi^1, \chi^2) = g_{1(0)}(\chi^1) \cos^2[\chi^2 - \chi_{(0)}^2(\chi^1)] \quad (4.17)$$

for $p_{(0)}(\chi^1) = 0$, where $g_{1(0)}(\chi^1)$, $\chi_{(0)}^2(\chi^1)$ and $p_{(0)}(\chi^1)$ are some functions of necessary smoothness class on variable $\chi^1 = x^1/\sqrt{\kappa\varepsilon}$, when ε is the energy density. If we consider $\Upsilon_{3i} = 0$ and a nontrivial diagonal components of energy-momentum d-tensor, $\Upsilon_\beta^\alpha = \text{diag}[0, 0, -\varepsilon]$, the N-connection coefficients $q_i(\chi^i, t)$ could be arbitrary functions.

For simplicity, in our further considerations we shall apply the solution (4.17).

The d-metric (4.7) with the coefficients (4.17) and (4.13) gives a general description of a class of solutions with generic local anisotropy of the Einstein equations (2.18).

Let us construct static black la-hole solutions for $s_{(0)}(\chi^k) = 0$ in (4.13).

In order to construct an explicit la-solution we have to chose some coefficients $h_{3(0)}(\chi^k)$, $g_{1(0)}(\chi^1)$ and $\chi_0(\chi^1)$ from some physical considerations. For instance, the Schwarzschild solution is selected from a general 4D metric with some general coefficients of static, spherical symmetry by relating the radial component of metric with the Newton gravitational potential. In this section, we construct a locally anisotropic BTZ like solution by supposing that it is conformally equivalent to the BTZ solution if one neglects anisotropies on angle θ ,

$$g_{1(0)}(\chi^1) = \left[r \left(-M_0 + \frac{r^2}{l^2} \right) \right]^{-2},$$

where $M_0 = \text{const} > 0$ and $-1/l^2$ is a constant (which is to be considered the cosmological from the locally isotropic limit. The time-time coefficient of d-metric is chosen

$$h_3(\chi^1, \chi^2) = r^{-2} \lambda_3(\chi^1, \chi^2) \cos^2[\chi^2 - \chi_{(0)}^2(\chi^1)]. \quad (4.18)$$

If we chose in (4.18)

$$\lambda_3 = \left(-M_0 + \frac{r^2}{l^2} \right)^2,$$

when the constant

$$r_h = \sqrt{M_0 l}$$

defines the radius of a circular horizon, the la-solution is conformally equivalent, with the factor $r^{-2} \cos^2[\chi^2 - \chi_{(0)}^2(\chi^1)]$, to the BTZ solution embedded into a anholonomic background given by arbitrary functions $q_i(\chi^i, t)$ which are defined by some initial conditions of gravitational la-background polarization.

A more general class of la-solutions could be generated if we put, for instance,

$$\lambda_3(\chi^1, \chi^2) = \left(-M(\theta) + \frac{r^2}{l^2} \right)^2,$$

with

$$M(\theta) = \frac{M_0}{(1 + e \cos \theta)^2},$$

where $e < 1$. This solution has a horizon, $\lambda_3 = 0$, parametrized by an ellipse

$$r = \frac{r_h}{1 + e \cos \theta}$$

with parameter r_h and eccentricity e .

We note that our solution with elliptic horizon was constructed for a diagonal energy-momentum d-tensor with nontrivial energy density but without cosmological constant. On the other hand the BTZ solution was constructed for a generic 3D cosmological constant. There is not a contradiction here because the la-solutions can be considered for a d-tensor $\Upsilon_{\beta}^{\alpha} = \text{diag}[p_1 - 1/l^2, p_2 - 1/l^2, -\varepsilon - 1/l^2]$ with $p_{1,2} = 1/l^2$ and $\varepsilon_{(eff)} = \varepsilon + 1/l^2$ (for $\varepsilon = \text{const}$ the last expression defines the effective constant r_a). The locally isotropic limit to the BTZ black hole could be realized after multiplication on r^2 and by approximations $e \simeq 0$, $\cos[\theta - \theta_0(\chi^1)] \simeq 1$ and $q_i(x^k, t) \simeq 0$.

4.2 Oscillating elliptic horizons

The simplest way to construct 3D solutions of the Einstein equations with oscillating in time horizon is to consider matter states with constant nonvanishing values of $\Upsilon_{31} = \text{const}$. In this case the coefficient h_3 could depend on t -variable. For instance, we can chose such initial values when

$$h_3(\chi^1, \theta, t) = r^{-2} \left(-M(t) + \frac{r^2}{l^2} \right) \cos^2[\theta - \theta_0(\chi^1)] \quad (4.19)$$

with

$$M = M_0 \exp(-\tilde{p}t) \sin \tilde{\omega}t,$$

or, for an another type of anisotropy,

$$h_3(\chi^1, \theta, t) = r^{-2} \left(-M_0 + \frac{r^2}{l^2} \right) \cos^2 \theta \sin^2[\theta - \theta_0(\chi^1, t)] \quad (4.20)$$

with

$$\cos \theta_0(\chi^1, t) = e^{-1} \left(\frac{r_a}{r} \cos \omega_1 t - 1 \right),$$

when the horizon is given parametrically,

$$r = \frac{r_a}{1 + e \cos \theta} \cos \omega_1 t,$$

where the new constants (comparing with those from the previous subsection) are fixed by some initial and boundary conditions as to be $\tilde{p} > 0$, and $\tilde{\omega}$ and ω_1 are treated as some real numbers.

For a prescribed value of $h_3(\chi^1, \theta, t)$ with non-zero source Υ_{31} , in the equation (4.6), we obtain

$$q_1(\chi^1, \theta, t) = \kappa \Upsilon_{31} \left(\frac{\partial^2}{\partial t^2} \ln |h_3(\chi^1, \theta, t)| \right)^{-1}. \quad (4.21)$$

A solution (4.1) of the Einstein equations (4.5) and (4.6) with $g_2(\chi^i) = 1$ and $g_1(\chi^1, \theta)$ and $h_3(\chi^1, \theta, t)$ given respectively by formulas (4.17) and (4.19) describe a 3D evaporating black la-hole solution with circular oscillating in time horizon. An another type of solution, with elliptic oscillating in time horizon, could be obtained if we choose (4.20). The non-trivial coefficient of the N-connection must be computed following the formula (4.21).

5 4D la-solutions

5.1 Basic properties

The purpose of this section is the construction of d-metrics which are conformally equivalent to some la-deformations of black hole, torus and cylinder like solutions in general relativity. We shall analyze 4D d-metrics of type

$$\delta s^2 = g_1(x^k) (dx^1)^2 + (dx^2)^2 + h_3(x^i, z) (\delta z)^2 + h_4(x^i, z) (\delta y^4)^2. \quad (5.1)$$

The Einstein equations (3.7) with the Ricci h-tensor (3.3) and diagonal energy momentum d-tensor transforms into

$$\frac{\partial^2 g_1}{\partial(x^2)^2} - \frac{1}{2g_1} \left(\frac{\partial g_1}{\partial x^2} \right)^2 - 2\kappa \Upsilon_3^3 g_1 = 0. \quad (5.2)$$

By introducing a dimensionless coordinate, $\chi^2 = x^2/\sqrt{|\kappa \Upsilon_3^3|}$, and the variable $p = g'_1/g_1$, where by 'prime' in this section is considered the partial derivative $\partial/\partial \chi^2$, the equation (5.2) transforms into

$$p' + \frac{p^2}{2} + 2\epsilon = 0, \quad (5.3)$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = 1(-1)$. The equations (5.2) and (5.3) are, correspondingly, equivalent to the equations (4.8) and (4.11) with that difference that in this section we are dealing with 4D coefficients and values. The solutions for the h-metric are parametrized like (4.14), (4.15), and (4.16) and the coefficient $g_1(\chi^i)$ is given by a similar to (4.17) formula (for simplicity, here we elucidate the case $\epsilon < 0$) which for $p_{(0)}(\chi^1) = 0$ transforms into

$$g_1(\chi^1, \chi^2) = g_{1(0)}(\chi^1) \cos^2[\chi^2 - \chi_{(0)}^2(\chi^1)], \quad (5.4)$$

where $g_1(\chi^1)$, $\chi_{(0)}^2(\chi^1)$ and $p_{(0)}(\chi^1)$ are some functions of necessary smoothness class on variable $\chi^1 = x^1/\sqrt{\kappa\varepsilon}$, ε is the energy density. The coefficients $g_1(\chi^1, \chi^2)$ (5.4) and $g_2(\chi^1, \chi^2) = 1$ define a h-metric. The next step is the construction of h-components of d-metrics, $h_a = h_a(\chi^i, z)$, for different classes of symmetries of anisotropies.

The system of equations (3.8) with the vertical Ricci d-tensor component (3.4) is satisfied by arbitrary functions

$$h_3 = a_3(\chi^i) \text{ and } h_4 = a_4(\chi^i). \quad (5.5)$$

For v-metrics depending on three coordinates (χ^i, z) the v-components of the Einstein equations transform into (3.11) which reduces to (3.12) for prescribed values of $h_3(\chi^i, z)$, and, inversely, to (3.14) if $h_4(\chi^i, z)$ is prescribed. For h-metrics being conformally equivalent to (5.5) (see transforms (3.15)) we are dealing to equations of type (3.16) with integral varieties (3.17).

5.2 Rotation Hypersurfaces Horizons

We proof that there are static black hole and cylindrical like solutions of the Einstein equations with horizons being some 3D rotation hypersurfaces. The space components of corresponding d-metrics are conformally equivalent to some locally anisotropic deformations of the spherical symmetric Schwarzschild and cylindrical Weyl solutions. We note that for some classes of solutions the local anisotropy is contained in non-perturbative anholonomic structures.

5.2.1 Rotation ellipsoid configuration

There two types of rotation ellipsoids, elongated and flattened ones. We examine both cases of such horizon configurations.

Elongated rotation ellipsoid coordinates:

An elongated rotation ellipsoid hypersurface is given by the formula [7]

$$\frac{\tilde{x}^2 + \tilde{y}^2}{\sigma^2 - 1} + \frac{\tilde{z}^2}{\sigma^2} = \tilde{\rho}^2, \quad (5.6)$$

where $\sigma \geq 1$ and $\tilde{\rho}$ is similar to the radial coordinate in the spherical symmetric case.

The space 3D coordinate system is defined

$$\tilde{x} = \tilde{\rho} \sinh u \sin v \cos \varphi, \quad \tilde{y} = \tilde{\rho} \sinh u \sin v \sin \varphi, \quad \tilde{z} = \tilde{\rho} \cosh u \cos v,$$

where $\sigma = \cosh u$, ($0 \leq u < \infty$, $0 \leq v \leq \pi$, $0 \leq \varphi < 2\pi$). The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv} = \tilde{\rho}^2 \left(\sinh^2 u + \sin^2 v \right), \\ g_{\varphi\varphi} &= \tilde{\rho}^2 \sinh^2 u \sin^2 v. \end{aligned} \quad (5.7)$$

Let us introduce a d-metric

$$\delta s^2 = g_1(u, v) du^2 + dv^2 + h_3(u, v, \varphi) (\delta t)^2 + h_4(u, v, \varphi) (\delta \varphi)^2, \quad (5.8)$$

where δt and $\delta \varphi$ are N-elongated differentials.

As a particular solution (5.4) for the h-metric we choose the coefficient

$$g_1(u, v) = \cos^2 v. \quad (5.9)$$

The $h_3(u, v, \varphi) = h_3(u, v, \tilde{\rho}(u, v, \varphi))$ is considered as

$$h_3(u, v, \tilde{\rho}) = \frac{1}{\sinh^2 u + \sin^2 v} \frac{\left[1 - \frac{r_g}{4\rho}\right]^2}{\left[1 + \frac{r_g}{4\rho}\right]^6}. \quad (5.10)$$

In order to define the h_4 coefficient solving the Einstein equations, for simplicity with a diagonal energy-momentum d-tensor for vanishing pressure we must

solve the equation (3.12) which transforms into a linear equation (3.13) if $\Upsilon_1 = 0$. In our case $s(u, v, \varphi) = \beta^{-1}(u, v, \varphi)$, where $\beta = (\partial h_4 / \partial \varphi) / h_4$, must be a solution of

$$\frac{\partial s}{\partial \varphi} + \frac{\partial \ln \sqrt{|h_3|}}{\partial \varphi} s = \frac{1}{2}.$$

After two integrations (see [6]) the general solution for $h_4(u, v, \varphi)$, is

$$h_4(u, v, \varphi) = a_4(u, v) \exp \left[- \int_0^\varphi F(u, v, z) dz \right], \quad (5.11)$$

where

$$F(u, v, z) = 1 / \{ \sqrt{|h_3(u, v, z)|} [s_{1(0)}(u, v) + \frac{1}{2} \int_{z_0(u, v)}^z \sqrt{|h_3(u, v, z)|} dz] \},$$

$s_{1(0)}(u, v)$ and $z_0(u, v)$ are some functions of necessary smooth class. We note that if we put $h_4 = a_4(u, v)$ the equations (3.8) are satisfied for every $h_3 = h_3(u, v, \varphi)$.

Every d-metric (5.8) with coefficients of type (5.9), (5.10) and (5.11) solves the Einstein equations (3.7)–(3.10) with the diagonal momentum d-tensor

$$\Upsilon_\beta^\alpha = \text{diag} [0, 0, -\varepsilon = -m_0, 0],$$

when $r_g = 2\kappa m_0$; we set the light constant $c = 1$. If we choose

$$a_4(u, v) = \frac{\sinh^2 u \sin^2 v}{\sinh^2 u + \sin^2 v}$$

our solution is conformally equivalent (if not considering the time-time component) to the hypersurface metric (5.7). The condition of vanishing of the coefficient (5.10) parametrizes the rotation ellipsoid for the horizon

$$\frac{\tilde{x}^2 + \tilde{y}^2}{\sigma^2 - 1} + \frac{\tilde{z}^2}{\sigma^2} = \left(\frac{r_g}{4} \right)^2,$$

where the radial coordinate is redefined via relation $\tilde{r} = \tilde{\rho} \left(1 + \frac{r_g}{4\tilde{\rho}} \right)^2$. After multiplication on the conformal factor

$$\left(\sinh^2 u + \sin^2 v \right) \left[1 + \frac{r_g}{4\tilde{\rho}} \right]^4,$$

approximating $g_1(u, v) = \cos^2 v \approx 1$, in the limit of locally isotropic spherical symmetry,

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = r_g^2,$$

the d-metric (5.8) reduces to

$$ds^2 = \left[1 + \frac{r_g}{4\tilde{\rho}}\right]^4 (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) - \frac{\left[1 - \frac{r_g}{4\rho}\right]^2}{\left[1 + \frac{r_g}{4\rho}\right]^2} dt^2$$

which is just the Schwarzschild solution with the redefined radial coordinate when the space component becomes conformally Euclidean.

So, the d-metric (5.8), the coefficients of N-connection being solutions of (3.9) and (3.10), describe a static 4D solution of the Einstein equations when instead of a spherical symmetric horizon one considers a locally anisotropic deformation to the hypersurface of rotation elongated ellipsoid.

Flattened rotation ellipsoid coordinates

In a similar fashion we can construct a static 4D black hole solution with the horizon parametrized by a flattened rotation ellipsoid [7],

$$\frac{\tilde{x}^2 + \tilde{y}^2}{1 + \sigma^2} + \frac{\tilde{z}^2}{\sigma^2} = \tilde{\rho}^2,$$

where $\sigma \geq 0$ and $\sigma = \sinh u$.

The space 3D special coordinate system is defined

$$\tilde{x} = \tilde{\rho} \cosh u \sin v \cos \varphi, \quad \tilde{y} = \tilde{\rho} \cosh u \sin v \sin \varphi, \quad \tilde{z} = \tilde{\rho} \sinh u \cos v,$$

where $0 \leq u < \infty$, $0 \leq v \leq \pi$, $0 \leq \varphi < 2\pi$.

The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv} = \tilde{\rho}^2 (\sinh^2 u + \cos^2 v), \\ g_{\varphi\varphi} &= \tilde{\rho}^2 \sinh^2 u \cos^2 v. \end{aligned}$$

In the rest the black hole solution is described by the same formulas as in the previous subsection but with respect to new canonical coordinates for flattened rotation ellipsoid.

5.2.2 Cylindrical, Bipolar and Toroidal Configurations

We consider a d-metric of type (5.1). As a coefficient for h-metric we choose $g_1(\chi^1, \chi^2) = (\cos \chi^2)^2$ which solves the Einstein equations (3.7). The energy momentum d-tensor is chosen to be diagonal, $\Upsilon^\alpha_\beta = \text{diag}[0, 0, -\varepsilon, 0]$ with $\varepsilon \simeq m_0 = \int m_{(lin)} dl$, where $\varepsilon_{(lin)}$ is the linear 'mass' density. The coefficient $h_3(\chi^i, z)$ will be chosen in a form similar to (5.10),

$$h_3 \simeq \left[1 - \frac{r_g}{4\tilde{\rho}}\right]^2 / \left[1 + \frac{r_g}{4\tilde{\rho}}\right]^6$$

for a cylindrical elliptic horizon. We parametrize the second v-component as $h_4 = a_4(\chi^1, \chi^2)$ when the equations (3.8) are satisfied for every $h_3 = h_3(\chi^1, \chi^2, z)$.

Cylindrical coordinates:

Let us construct a solution of the Einstein equation with the horizon having the symmetry of ellipsoidal cylinder given by hypersurface formula [7]

$$\frac{\tilde{x}^2}{\sigma^2} + \frac{\tilde{y}^2}{\sigma^2 - 1} = \rho_*^2, \quad \tilde{z} = \tilde{z},$$

where $\sigma \geq 1$. The 3D radial coordinate \tilde{r} is to be computed from $\tilde{r}^2 = \rho_*^2 + \tilde{z}^2$.

The 3D space coordinate system is defined

$$\tilde{x} = \rho_* \cosh u \cos v, \quad \tilde{y} = \rho_* \sinh u \sin v \sin, \quad \tilde{z} = \tilde{z},$$

where $\sigma = \cosh u$, ($0 \leq u < \infty$, $0 \leq v \leq \pi$).

The hypersurface metric is

$$g_{uu} = g_{vv} = \rho_*^2 (\sinh^2 u + \sin^2 v), \quad g_{zz} = 1. \quad (5.12)$$

A solution of the Einstein equations with singularity on an ellipse is given by

$$\begin{aligned} h_3 &= \frac{1}{\rho_*^2 (\sinh^2 u + \sin^2 v)} \times \frac{\left[1 - \frac{r_g}{4\rho}\right]^2}{\left[1 + \frac{r_g}{4\rho}\right]^6}, \\ h_4 &= a_4 = \frac{1}{\rho_*^2 (\sinh^2 u + \sin^2 v)}, \end{aligned}$$

where $\tilde{r} = \tilde{\rho} \left(1 + \frac{r_g}{4\rho}\right)^2$. The condition of vanishing of the time-time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\frac{\tilde{x}^2}{\sigma^2} + \frac{\tilde{y}^2}{\sigma^2 - 1} = \left(\frac{\rho_{*(g)}}{4}\right)^2, \quad \tilde{z} = \tilde{z},$$

where $\rho_{*(g)} = 2\kappa m_{(lin)}$.

By multiplying the d-metric on the conformal factor

$$\rho_*^2 (\sinh^2 u + \sin^2 v) \left[1 + \frac{r_g}{4\tilde{\rho}}\right]^4,$$

where $r_g = \int \rho_{*(g)} dl$ (the integration is taken along the ellipse), for $\rho_* \rightarrow 1$, in the local isotropic limit, $\sin v \approx 0$, the space component transforms into (5.12).

Bipolar coordinates:

Let us construct 4D solutions of the Einstein equation with the horizon having the symmetry of the bipolar hypersurface given by the formula [7]

$$\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \frac{\tilde{\rho}}{\tan \sigma}\right)^2 + \tilde{z}^2 = \frac{\tilde{\rho}^2}{\sin^2 \sigma},$$

which describes a hypersurface obtained under the rotation of the circles

$$\left(\tilde{y} - \frac{\tilde{\rho}}{\tan \sigma}\right)^2 + \tilde{z}^2 = \frac{\tilde{\rho}^2}{\sin^2 \sigma}$$

around the axes Oz ; because $|c \tan \sigma| < |\sin \sigma|^{-1}$, the circles intersect the axes Oz . The 3D space coordinate system is defined

$$\begin{aligned}\tilde{x} &= \frac{\tilde{\rho} \sin \sigma \cos \varphi}{\cosh \tau - \cos \sigma}, & \tilde{y} &= \frac{\tilde{\rho} \sin \sigma \sin \varphi}{\cosh \tau - \cos \sigma}, \\ \tilde{z} &= \frac{\tilde{r} \sinh \tau}{\cosh \tau - \cos \sigma} \quad (-\infty < \tau < \infty, 0 \leq \sigma < \pi, 0 \leq \varphi < 2\pi).\end{aligned}$$

The hypersurface metric is

$$g_{\tau\tau} = g_{\sigma\sigma} = \frac{\tilde{\rho}^2}{(\cosh \tau - \cos \sigma)^2}, \quad g_{\varphi\varphi} = \frac{\tilde{\rho}^2 \sin^2 \sigma}{(\cosh \tau - \cos \sigma)^2}. \quad (5.13)$$

A solution of the Einstein equations with singularity on a circle is given by

$$h_3 = \left[1 - \frac{r_g}{4\tilde{\rho}}\right]^2 / \left[1 + \frac{r_g}{4\tilde{\rho}}\right]^6 \quad \text{and} \quad h_4 = a_4 = \sin^2 \sigma,$$

where $\tilde{r} = \tilde{\rho} \left(1 + \frac{r_g}{4\tilde{\rho}}\right)^2$. The condition of vanishing of the time-time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \frac{r_g}{2} c \tan \sigma\right)^2 + \tilde{z}^2 = \frac{r_g^2}{4 \sin^2 \sigma},$$

where $r_g = \int \rho_{*(g)} dl$ (the integration is taken along the circle), $\rho_{*(g)} = 2\kappa m_{(lin)}$.

By multiplying the d-metric on the conformal factor

$$\frac{1}{(\cosh \tau - \cos \sigma)^2} \left[1 + \frac{r_g}{4\tilde{\rho}}\right]^4, \quad (5.14)$$

for $\rho_* \rightarrow 1$, in the local isotropic limit, $\sin v \approx 0$, the space component transforms into (5.13).

Toroidal coordinates:

Let us consider solutions of the Einstein equations with toroidal symmetry of horizons. The hypersurface formula of a torus is [7]

$$\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \tilde{\rho} c \tanh \sigma\right)^2 + \tilde{z}^2 = \frac{\tilde{\rho}^2}{\sinh^2 \sigma}.$$

The 3D space coordinate system is defined

$$\begin{aligned}\tilde{x} &= \frac{\tilde{\rho} \sinh \sigma \cos \varphi}{\cosh \tau - \cos \sigma}, & \tilde{y} &= \frac{\tilde{\rho} \sin \sigma \sin \varphi}{\cosh \tau - \cos \sigma}, \\ \tilde{z} &= \frac{\tilde{\rho} \sinh \sigma}{\cosh \tau - \cos \sigma} \quad (-\pi < \sigma < \pi, 0 \leq \tau < \infty, 0 \leq \varphi < 2\pi).\end{aligned}$$

The hypersurface metric is

$$g_{\sigma\sigma} = g_{\tau\tau} = \frac{\tilde{\rho}^2}{(\cosh \tau - \cos \sigma)^2}, g_{\varphi\varphi} = \frac{\tilde{\rho}^2 \sin^2 \sigma}{(\cosh \tau - \cos \sigma)^2}. \quad (5.15)$$

This, another type of solution of the Einstein equations with singularity on a circle, is given by

$$h_3 = \left[1 - \frac{r_g}{4\tilde{\rho}} \right]^2 / \left[1 + \frac{r_g}{4\tilde{\rho}} \right]^6 \text{ and } h_4 = a_4 = \sinh^2 \sigma,$$

where $\tilde{r} = \tilde{\rho} \left(1 + \frac{r_g}{4\rho} \right)^2$. The condition of vanishing of the time-time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \frac{r_g}{2 \tanh \sigma} c \right)^2 + \tilde{z}^2 = \frac{r_g^2}{4 \sinh^2 \sigma},$$

where $r_g = \int \rho_{*(g)} dl$ (the integration is taken along the circle), $\rho_{*(g)} = 2\kappa m_{(lin)}$.

By multiplying the d-metric on the conformal factor (5.14), for $\rho_* \rightarrow 1$, in the local isotropic limit, $\sin v \approx 0$, the space component transforms into (5.15).

5.3 A Schwarzschild like la-solution

The d-metric of type (5.8) is taken

$$\delta s^2 = g_1(\chi^1, \theta) d(\chi^1)^2 + d\theta^2 + h_3(\chi^1, \theta, \varphi) (\delta t)^2 + h_4(\chi^1, \theta, \varphi) (\delta \varphi)^2, \quad (5.16)$$

where on the horizontal subspace $\chi^1 = \rho/r_a$ is the dimensionless radial coordinate (the constant r_a will be defined below), $\chi^2 = \theta$ and in the vertical subspace $y^3 = z = t$ and $y^4 = \varphi$. The energy-momentum d-tensor is taken to be diagonal $\Upsilon^\alpha_\beta = \text{diag}[0, 0, -\varepsilon, 0]$. The coefficient g_1 is chosen to be a solution of type (5.4)

$$g_1(\chi^1, \theta) = \cos^2 \theta.$$

For

$$h_4 = \sin^2 \theta \text{ and } h_3(\rho) = -\frac{[1 - r_a/4\rho]^2}{[1 + r_a/4\rho]^6},$$

where $r = \rho \left(1 + \frac{r_g}{4\rho} \right)^2$, $r^2 = x^2 + y^2 + z^2$, $r_a \doteq r_g$ is the Schwarzschild gravitational radius, the d-metric (5.16) describes a la-solution of the Einstein equations which is conformally equivalent, with the factor $\rho^2 (1 + r_g/4\rho)^2$, to the Schwarzschild solution (written in coordinates $(\rho, \theta, \varphi, t)$), embedded into a la-background given by non-trivial values of $q_i(\rho, \theta, t)$ and $n_i(\rho, \theta, t)$. In the anisotropic case we can extend the solution for anisotropic (on angle θ) gravitational polarizations of point particles masses, $m = m(\theta)$, for instance in elliptic form, when

$$r_a(\theta) = \frac{r_g}{(1 + e \cos \theta)}$$

induces an ellipsoidal dependence on θ of the radial coordinate,

$$\rho = \frac{r_g}{4(1 + e \cos \theta)}.$$

We can also consider arbitrary solutions with $r_a = r_a(\theta, t)$ of oscillation type, $r_a \simeq \sin(\omega_1 t)$, or modelling the mass evaporation, $r_a \simeq \exp[-st]$, $s = \text{const} > 0$.

So, fixing a physical solution for $h_3(\rho, \theta, t)$, for instance,

$$h_3(\rho, \theta, t) = -\frac{[1 - r_a \exp[-st]/4\rho(1 + e \cos \theta)]^2}{[1 + r_a \exp[-st]/4\rho(1 + e \cos \theta)]^6},$$

where $e = \text{const} < 1$, and computing the values of $q_i(\rho, \theta, t)$ and $n_i(\rho, \theta, t)$ from (3.9) and (3.10), corresponding to given h_3 and h_4 , we obtain a local generalization of the Schwarzschild metric.

We note that fixing this type of anisotropy, in the locally isotropic limit we obtain not just the Schwarzschild metric but a conformally transformed one, multiplied on the factor $1/\rho^2(1 + r_g/4\rho)^4$.

6 Final remarks

We have presented new classes of three and four dimensional black hole solutions with local anisotropy which are given both with respect to a coordinate basis or to an anholonomic frame defined by a N-connection structure. We proved that for a corresponding ansatz such type of solutions can be imbedded into the usual (three or four dimensional) Einstein gravity. It was demonstrated that in general relativity there are admitted static, but anisotropic (with nonspheric symmetry), and elliptic oscillating in time black hole like configurations with horizons of events being elliptic (in three dimensions) and rotation ellipsoidal, elliptic cylinder, toroidal and another type of closed hypersurfaces or cylinders.

From the results obtained, it appears that the components of metrics with generic local anisotropy are somehow undetermined from field equations if the type of symmetry and a correspondence with locally isotropic limits are not imposed. This is the consequence of the fact that in general relativity only a part of components of the metric field (six from ten in four dimensions and three from six in three dimensions) can be treated as dynamical variables. This is caused by the Bianchi identities which hold on (pseudo) Riemannian spaces. The rest of components of metric should be defined from some symmetry prescriptions on the type of locally anisotropic solutions and corresponding anholonomic frames and, if existing, compatibility with the locally isotropic limits when some physically motivated coordinate and/or boundary conditions are enough to state and solve the Cauchy problem.

Some of the problems discussed so far might be solved by considering theories containing non-trivial torsion fields like metric-affine and gauge gravity

and for so-called generalized Finsler–Kaluza–Klein models. More general solutions connected with locally anisotropic low energy limits in string/M-theory and supergravity could be also generated by applying the method of computation with respect to anholonomic (super) frames adapted to a N-connection structure. This topic is currently under study.

Acknowledgments

The author would like to thank P J Steinhardt, N G Turok and V A Rubakov for support of his participation at NATO ASI "Structure Formation in the Universe", July 26 – August 6, 1999 (held at Isaac Newton Institute for Mathematical Sciences, Cambridge University, UK), where this work was communicated. He is also very grateful to H Dehnen for hospitality and useful discussions during his visit at Konstanz University, Germany.

References

- [1] Bañados M, Teitelboim C and Zanelli J 1992 *Phys. Rev. Lett.* **69** 1849
Bañados M, Henneaux M, Teitelboim C and Zanelli J 1993 *Phys. Rev. D* **48** 1506
- [2] Barthel W 1963 *J. Reine Angew. Math.* **212** 120
- [3] Cartan E 1934 *Les Espaces de Finsler* (Herman, Paris)
- [4] Hawking S W and Ellis C F R 1973 *The Large Scale Structure of Spacetime* (Cambridge University Press)
- [5] Hehl F M, Mc Grea J D, Mielke E W, and Ne'eman Y 1995 *Phys. Rep.* **258** 1
Tseytlin A A 1982 *Phys. Rev. D* **26** 3327
Ponomarev V N, Barvinsky A O, and Obukhov Yu N 1985 *Geometrodynamical Methods and the Gauge Approach to the Theory of Gravitational Interactions* (Energoatomizsat, Moscow) [in Russian]
- [6] Kamke E, *Differential Gleichungen, Lösungsmethoden und Lösungen: I. Gewöhnliche Differentialgleichungen* (Lipzig, 1959).
- [7] Korn G A and Korn T M, *Mathematical Handbook* (McGraw–Hill Book Company, 1968)
- [8] Miron R and Anastasiei M 1994 *The Geometry of Lagrange Spaces: Theory and Applications* (Kluwer Academic Publishers, Dordrecht, Boston, London)
- [9] Vacaru S 1996 *J. Math. Phys.* **37** 508
Vacaru S 1998 *J. High Energy Physics* **09** 011, hep-th / 9807214

- [10] Vacaru S 1997 *Ann. Phys. (NY)* **256** 39, gr-qc /9604013
Vacaru S 1997 *Nucl. Phys.* **B434** 590, hep-th / 9611034
- [11] Vacaru S and Goncharenko Yu 1995 *Int. J. Theor. Phys.* **34** 1955
Vacaru S 1998 *Gauge Gravity and Conservation Laws in Higher Order Anisotropic Spaces*, hep-th / 9810229.